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On the Limiting Behavior of Burger's Equation

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The limiting behavior as the viscosity goes to zero of the solution of the first boundary value problem for Burger's equation is considered. The method consists in identifying the solution of Burger's equation with the optimal control of an appropriate stochastic control problem.

In this note we study the limiting behavior as the viscosity goes to zero of the solution to the first boundary value problem for Burger's equation. Our approach consists in identifying the solution of Burger's equation with the optimal control of an appropriate stochastic control problem. Using this approach we extend the recent work of Bui An Ton [1]; see Remark, Section 3. Our approach was first used by Fleming [2] to treat the Cauchy problem for Burger's equation; this problem has also been discussed by Hopf [3].

1

For $\epsilon \geqslant 0$ consider Burger's equation

$$-v_t^\epsilon + \epsilon v_{xx}^\epsilon - v^\epsilon v_x^\epsilon + f = 0 \tag{1}$$

on $(a, b) \times (0, T)$ subject to the boundary data

$$\begin{aligned} v^\epsilon(a, t) = v^\epsilon(b, t) &= 0, \qquad 0 \leqslant t \leqslant T; \\ v^\epsilon(x, 0) &= 0, \qquad a \leqslant x \leqslant b. \end{aligned} \tag{2}$$

Assume that f is a function of class C^∞ on $[a, b] \times [0, T]$ satisfying

$$f(a, t) = (\partial^n f/\partial x^n)(a, t) = f(b, t) = (\partial^n f/\partial x^n)(b, t) = 0 \tag{3}$$

for $0 \leqslant t \leqslant T$ and n a positive integer. Then we have the following theorem.

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THEOREM. *Let $f_x(x, t) > -M$ on $[a, b] \times [0, T]$ for some positive constant M. Then there exists R with $\pi/2M^{1/2} \leq R \leq T$ such that the following hold.*

(i) *There exists v^0 satisfying* (1) *and* (2) *with $\epsilon = 0$ such that v^0 is a C^∞ function on $[a, b] \times [0, R]$.*

(ii) *There exist functions $\chi_1, \chi_2, \ldots$ such that for any positive integer k,*

$$v^\epsilon = v^0 + \sum_{i=1}^{k} \epsilon^i \chi_i + o(\epsilon^k) \tag{4}$$

uniformly on $[a, b] \times [0, R]$.

Proof. We utilize the approach of [2] and Theorem 7.1 therein. Define a function F on $(-\infty, \infty) \times [0, T]$ by the formulas

$$\begin{aligned} F(x, t) &= \int_{(a+b)/2}^{x} f(y, t)\, dy, \qquad & a \leq x \leq b, \\ F(x, t) &= F(2b - x, t), \qquad & b \leq x \leq 2b - a, \end{aligned} \tag{5}$$

and $F(x, t)$ is, for fixed t, a function of class C^∞, periodic with period $2(b - a)$.

Now consider the completely observable stochastic control problem with control u, state equations

$$d\xi = u\, dt + (2\epsilon)^{1/2}\, dw, \tag{6}$$

where w is one dimensional Brownian motion, initial condition $\xi(s) = x$, and cost function

$$E \int_s^T F(x, t) + (u^2/2)\, dt. \tag{7}$$

Let $\phi^\epsilon(x, s)$ denote the minimal cost over the class of feedback controls. From [2] we have that, for $\epsilon > 0$, $\phi^\epsilon(x, s)$ is a solution of class C^∞ on $(-\infty, \infty) \times [0, T]$ of the equation

$$\epsilon\phi^\epsilon_{xx} + \phi_t^\epsilon - (\phi_x^\epsilon)^2/2 + F = 0 \tag{8}$$

with boundary condition $\phi^\epsilon(x, T) = 0$.

For $\epsilon = 0$ we have the corresponding deterministic control problem. Suppose, for the moment, that there exists a positive constant Q such that there is a unique optimal control for the deterministic control problem for each initial state with initial time s satisfying $T - Q \leq s \leq T$. Then ϕ^0 is a solution of class C^∞ of (8) on $(-\infty, \infty) \times [T - Q, T]$ satisfying $\phi^0(x, T) = 0$.

Using the periodicity assumptions on F, and state equations (6) and cost function (7), we have that $\phi_x^\epsilon(a, t) = \phi_x^\epsilon(b, t) = 0$ for $\epsilon > 0$ and $T - Q < t \leq T$. Further, $\phi_x^\epsilon(x, T) = 0$. Then

$$v^\epsilon(x, t) = \phi_x^\epsilon(x, T - t), \qquad 0 \leq t \leq Q, \quad a \leq x \leq b, \tag{9}$$

and the correspondence between the solution of Burger's equation and the control problem has been made. Let us continue with our momentary assumption concerning the unique optimal control for the deterministic control problem. Then [2, Theorem 7.1] shows that there exist functions $\theta_1, \theta_2, \ldots$ such that for any positive integer k,

$$\phi_x^\epsilon(x, T - t) = \phi_x^0(x, T - t) + \sum_{j=1}^{k} \epsilon^j(\theta_j)_x(x, T - t) + o(\epsilon^k) \tag{10}$$

uniformly on $[a, b] \times [T - Q, T]$. Equation (4) then follows on $[a, b] \times [0, Q]$ with identification (9) and $\chi_j(x, t) = (\theta_j)_x(x, T - t)$.

To complete the proof of the theorem we must remove our momentary assumption. Thus we need to show that there exists $Q > \pi/2M^{1/2}$ such that there is a unique optimal control for each initial state with initial time s satisfying $T - Q \leq s \leq T$.

If u^0 is an optimal open loop control corresponding to the initial point $\xi(s) = x$, then $u^0(t) = -p(t)$ where $\xi(t), p(t)$ satisfy the equations

$$\begin{aligned} d\xi/dt &= -p(t), \\ dp/dt &= -f(\xi, t), \end{aligned} \tag{11}$$

with boundary conditions $\xi(s) = x$, $p(T) = 0$. Consider Eq. (11) with boundary conditions $\xi(T, \alpha) = \alpha$, $p(T, \alpha) = 0$. If the optimal control for $\xi(s) = x$ is not unique, then there exist α, α', $\alpha \neq \alpha'$, such that the solutions $\xi(t, \alpha)$, $p(t, \alpha)$ and $\xi(t, \alpha')$, $p(t, \alpha')$ both satisfy (11) and the additional boundary conditions $\xi(s, \alpha) = x$, $\xi(s, \alpha') = x$.

Let us show the existence of α, α', $\alpha \neq \alpha'$, is impossible if $T - (\pi/2M^{1/2}) < s \leq T$. Now $(\partial\xi/\partial\alpha)(t, \alpha)$, $(\partial p/\partial\alpha)(t, \alpha)$ satisfy

$$\begin{aligned} (d/dt)((\partial\xi/\partial\alpha)(t, \alpha)) &= -(\partial p/\partial\alpha)(t, \alpha), \\ (d/dt)((\partial p/\partial\alpha)(t, \alpha)) &= -f_x(\xi(t, \alpha), t)(\partial\xi/\partial\alpha)(t, \alpha), \end{aligned} \tag{12}$$

with boundary conditions $(\partial\xi/\partial\alpha)(T, \alpha) = 1$, $(\partial p/\partial\alpha)(T, \alpha) = 0$. A simple estimate shows that $(\partial\xi/\partial\alpha)(s, \alpha) \neq 0$ for $T - (\pi/2M^{1/2}) < s \leq T$ and hence that $\xi(s, \alpha) \neq \xi(s, \alpha')$. This gives the uniqueness of the optimal control on $T - (\pi/2M^{1/2}) < s \leq T$ and thus completes the proof of the theorem.

2

In this section we illustrate the calculation of the functions $\phi_x{}^0$, $(\theta_j)_x$, and consequently of the functions v^0, χ_j. Let $\xi(t, \alpha)$, $p(t, \alpha)$ be as previously defined. Then, by the method of characteristics,

$$\phi_x{}^0(\xi(t, \alpha), t) = p(t, \alpha), \tag{13}$$

and hence $v^0(\xi(t, \alpha), T - t) = p(t, \alpha)$.

The calculation of $(\theta_1)_x$ is slightly more complicated. Let $q(t) = \theta_1(\xi(t, \alpha), t)$ and $r(t) = (\theta_1)_x (\xi(t, \alpha), t)$. Then $q(t)$, $r(t)$ satisfy

$$\begin{aligned} dq/dt &= -\phi^0_{xx}(\xi(t, \alpha), t), \\ dr/dt &= \phi^0_{xx}(\xi(t, \alpha), t)\, q(t) - \phi^0_{xxx}(\xi(t, \alpha), t), \end{aligned} \tag{14}$$

with boundary conditions $q(T) = r(T) = 0$. The coefficients $\phi^0_{xx}(\xi(t, \alpha), t)$, $\phi^0_{xxx}(\xi(t, \alpha), t)$ in (14) are found from the relations

$$\begin{aligned} \phi^0_{xx}(\xi(t, \alpha), t) \frac{\partial \xi}{\partial \alpha}(t, \alpha) &= \frac{\partial p}{\partial \alpha}(t, \alpha), \\ \phi^0_{xxx}(\xi(t, \alpha), t) \left[\frac{\partial \xi}{\partial \alpha}(t, \alpha)\right]^2 + \phi^0_{xx}(\xi(t, \alpha), t) \frac{\partial^2 \xi}{\partial \alpha^2}(t, \alpha) &= \frac{\partial^2 p}{\partial \alpha^2}(t, \alpha), \end{aligned} \tag{15}$$

where $(\partial \xi/\partial \alpha)(t, \alpha)$, $(\partial p/\partial \alpha)(t, \alpha)$, $(\partial^2 \xi/\partial \alpha^2)(t, \alpha)$, $(\partial^2 p/\partial \alpha^2)(t, \alpha)$ are found from the differential equations (12) and the differential equations

$$\begin{aligned} (d/dt)\,((\partial^2 \xi/\partial \alpha^2)(t, \alpha)) &= -(\partial^2 p/\partial \alpha^2)(t, \alpha), \\ (d/dt)\,((\partial^2 p/\partial \alpha^2)(t, \alpha)) &= -f_x(\xi(t, \alpha), t)\,(\partial^2 \xi/\partial \alpha^2)(t, \alpha) \\ &\quad - f_{xx}(\xi(t, \alpha), t)\,[(\partial \xi/\partial \alpha)(t, \alpha)]^2, \end{aligned} \tag{16}$$

with boundary conditions $(\partial^2 \xi/\partial \alpha^2)(T, \alpha) = (\partial^2 p/\partial \alpha^2)(T, \alpha) = 0$. The calculation of $(\theta_j)_x$, $j \geq 2$, and hence χ_j, $j \geq 2$, is found in a similar but more tedious manner. See [2] or [4] for details.

3

Remark. For expansion (4) to be valid for $k = 1$, then f need only be C^2 and (3) hold for $n = 1, 2$. This should be contrasted with one of Bui An Ton's results. Using standard notation, let f be an element of $L^\infty(0, T; W_0^{3,2}(a, b))$.

Then he shows that there exists a positive τ (not explicitly computed) such that

$$\left(\int_a^b |v^\epsilon(x,t) - v^0(x,t)|^2\,dx\right)^{1/2} = O(\epsilon^{1/2})$$

uniformly on $(0, \tau)$.

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